Aufgaben Math. I ET WS 13/14

1a) $2\vec{a}-\vec{b}=(4,-3,1)$; b) $b(\vec{c}+2\vec{a})=-4$ c) $|\vec{a}+2\vec{b}|=|(-3,-3,3)|=3\sqrt{3}$

d) $\vec{a}\times\vec{c}=(-2,1,3)$ e) $\alpha_1=90^\circ$ $\alpha_2\approx 61{,}87^\circ$

2 a) $F=|\vec{a}\times\vec{b}|=|(t+3,1,t-2)|=\sqrt{2t^2+2t+14}=\sqrt{2}\sqrt{(t+\frac{1}{2})^2+\frac{27}{4}}$
minimal für $t=-\frac{1}{2}$

b) $\vec{a},\vec{b},\vec{c}$ linear unabh. $\Leftrightarrow [\vec{a},\vec{b},\vec{c}]=\begin{vmatrix}-1 & 1 & -1\\ t & 2 & t+1\\ 1-t & t+2 & 1\end{vmatrix}=\begin{vmatrix}-1 & 0 & 0\\ t & t+2 & 1\\ 1-t & 3 & t\end{vmatrix}\neq 0$

$\Leftrightarrow t^2+2t-3\neq 0 \Leftrightarrow t\in\mathbb{R}\setminus\{1;-3\}$

3.) a) $P(3)=340$, $P(2)=P(-2)=P(-1)=0$, $P(1)=-30$

b) $P(x)=(x-2)(x+2)(x+1)(x^2+2x+2)$

4) a) $f'(x)=\arctan x+\frac{x}{1+x^2}+e^x$ $D_f=D_{f'}=\mathbb{R}$

b) $f=\frac{1}{4}(3\ln x+x)+2\ln|\sin x|$ $f'(x)=\frac{3}{4}\frac{1}{x}+\frac{1}{4}+2\cot x$, $D_f=D_{f'}=\mathbb{R}^+\setminus\{k\pi\}$, $k\in\mathbb{Z}$.

c) $f'=\frac{1}{\cos^2 x}\,2^{\tan x}$ $D_{f'}=D_f=\mathbb{R}-\{(k+\frac{1}{2})\pi,\ k\in\mathbb{Z}\}$

d) $f(x)=\cosh(\operatorname{arsinh} x^2)=\sqrt{1+\sinh^2(\operatorname{arsinh} x^2)}=\sqrt{1+x^4}$

$\Rightarrow f'(x)=\frac{2x^3}{\sqrt{1+x^4}}$ $D_{f'}=D_f=\mathbb{R}$.

e) $f'=\frac{1}{\sqrt{1-\cos^2 x}}(-\sin x)=-\frac{\sin x}{|\sin x|}=-\operatorname{sign}(\sin x)$ $x\neq k\pi$

$D_f=\mathbb{R}$, $D_{f'}=\mathbb{R}\setminus\{k\pi,\ k\in\mathbb{Z}\}$

f) $f(x)=\frac{1}{2}(e^{\ln x^4}-e^{-\ln x^4})=\frac{1}{2}(x^4-x^{-4})$ $f'=2(x^3+x^{-5})$

$D_{f'}=D_f=\mathbb{R}\setminus\{0\}$

g) $f(x)=x^x=e^{x\ln x}$ $f'(x)=x^x(\ln x+1)$ $D_{f'}=D_f=\mathbb{R}^+$

5) $\ln(1+x)\geq x-\frac{x^2}{2} \Leftrightarrow f(x)=\ln(1+x)-x+\frac{x^2}{2}\geq 0$ für $x\geq 0$

$f(0)=0$, $f'(x)=\frac{1}{1+x}-1+x=\frac{x^2}{1+x}>0$ für $x>0 \Rightarrow f$ für $x\geq 0$

streng monoton wachsend. $\Rightarrow f(x)\geq 0$ für $x\geq 0$.

6) a) $\lim\limits_{x\to 0}\frac{\cos(3x)-1}{\sinh x^2}\ (\frac{0}{0})=\lim\limits_{x\to 0}\frac{-3\sin(3x)}{2x\cosh x^2}\ (\frac{0}{0})=\lim\limits_{x\to 0}\frac{-9\cos(3x)}{2\cosh x^2+4x^2\sinh x^2}=-\frac{9}{2}$

b) $\lim\limits_{x\to+\infty}\frac{\arctan e^x}{x}=0$, da $0<\arctan e^x<\frac{\pi}{2}$, $x\to+\infty$

c) $\lim\limits_{x\to+\infty}\frac{e^x}{\cosh x}=\lim\limits_{x\to+\infty}2\frac{e^x}{e^x+e^{-x}}=2\lim\limits_{x\to+\infty}\frac{1}{1+e^{-2x}}=2$

d) $\lim\limits_{x\to 1}(\sin^2(\pi x))^{x^2-1}\ (0^0)=e^{\lim\limits_{x\to 1}(x^2-1)2\ln|\sin(\pi x)|}\ (0\cdot\infty)=e^{2\lim\limits_{x\to 1}\underbrace{(x+1)}_{2}\frac{\ln|\sin\pi x|}{\frac{1}{x-1}}}\ (\frac{\infty}{\infty})$

$=e^{4\lim\limits_{x\to 1}\frac{-\pi\cos(\pi x)(x-1)^2}{\sin(\pi x)}}=e^{4\pi\lim\limits_{x\to 1}\frac{(x-1)^2}{\sin\pi x}}\ (\frac{0}{0})=e^{8\lim\limits_{x\to 1}\frac{x-1}{\cos\pi x}}=e^0=1$

d) $\lim\limits_{x\to 0}(\cos x)^{\sin^{-2}x}\ (1^\infty)=e^{\lim\limits_{x\to 0}\frac{\ln\cos x}{\sin^2 x}}\ (\frac{0}{0})=e^{\lim\limits_{x\to 0}\frac{-\sin x}{\cos x\cdot 2\sin x\cos x}}=e^{-\frac{1}{2}}$

e) $\lim\limits_{x\to 0}\frac{1}{x}\left(\cot x-\frac{1}{\sin x}\right)\ (\infty-\infty)=\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{\cos x}{\sin x}-\frac{1}{\sin x}\right)=\lim\limits_{x\to 0}\frac{\cos x-1}{x\sin x}\ (\frac{0}{0})=$

$=\lim\limits_{x\to 0}\frac{-\sin x}{\sin x+x\cos x}\ (\frac{0}{0})=\lim\frac{-\cos x}{\ldots}$

6 g) $\lim\limits_{x\to\frac{\pi}{2}} (\tan^2 x)^{\cos x} \quad (\infty^0) = e^{\lim\limits_{x\to\frac{\pi}{2}} \cos x \cdot 2\ln|\tan x|} \quad (0\cdot\infty)$

$= e^{\lim\limits_{x\to\frac{\pi}{2}} 2 \frac{\ln|\tan x|}{\frac{1}{\cos x}}} \quad (\frac{\infty}{\infty}) = e^{2\lim\limits_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}\cdot\cot x}{-\frac{1}{\cos^2 x}\sin x}} = e^{-2\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos x}{\sin^2 x}} = e^0 = 1$

7) Schnittpunkte: $(0,1)$; $(2,0)$
Schnittwinkel: $\alpha_1 \approx 1{,}2925$ $\alpha_2 \approx 1{,}2855$

8) $P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2$

$f(4) = 0 \quad f'(x) = \frac{1}{1+x^2} \Rightarrow f'(4) = \frac{1}{17}, \; f''(x) = -\frac{2x}{(1+x^2)^2}, \; f''(4) = -\frac{8}{17^2}$

$\Rightarrow P_2(x) = \frac{1}{17}(x-4) - \frac{4}{289}(x-4)^2$

$\Delta(x) = |R_2(x)| = \left|\frac{f'''(4+\delta(x-4))}{6}(x-4)^3\right| \quad 0<\delta<1 \quad |x-4| \le 1$

$f'''(x) = \frac{3x^2-1}{(1+x^2)^3} \Rightarrow \Delta(x) \le \frac{1}{3}\,\frac{74}{10^3} < 0{,}03$

9) $\sum\limits_{n=2}^{\infty} \frac{1}{n^2-n} = \lim\limits_{m\to\infty} \sum\limits_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \lim\limits_{m\to\infty}\left(1 - \frac{1}{m}\right) = 1$

10 a) wegen $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \frac{k+1}{k}\cdot\frac{2^k}{2^{k+1}} = \frac{1}{2} < 1 \Rightarrow$ absolute konv.

b) $\cos\pi k = (-1)^k \Rightarrow \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{k+2}$ alternierend $a_k = \frac{1}{k+2}$

$\Rightarrow \lim\limits_{k\to\infty} a_k = 0 \qquad a_{k+1} = \frac{1}{k+3} < \frac{1}{k+2} = a_k \Rightarrow a_k$ monotone

Nullfolge $\Rightarrow$ (Leibniz) Reihe konvergent.

c) $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \frac{\binom{2k+2}{k+1}}{(k+1)!}\,\frac{k!}{\binom{2k}{k}} = \lim\limits_{k\to\infty} \frac{(2k+2)!\,(k!)^3}{[(k+1)!]^3(2k)!} =$

$\lim\limits_{k\to\infty} \frac{(2k+1)}{(k+1)^2} = 0 \Rightarrow$ Absolute konvergenz

11) a) $r = \lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{n-1}{n(n+1)}\cdot\frac{(n+1)(n+2)}{n} = 1 \quad B = \{x \mid |x| < 1\}$

b) $r = \lim\limits_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim\limits_{k\to\infty} \frac{k}{k+1} = 1 \quad B = \{x \mid |x-4| < 1\}$

c) $a_{k^2} = \frac{1}{(k+1)!} \Rightarrow$ wegen $1 \le \sqrt[k^2]{(k+1)!} \le \sqrt[k^2]{(k+1)^k} = \sqrt[k]{k+1}$

$\Rightarrow \lim\limits_{j\to\infty} \sqrt[j]{|a_j|} = 1 \quad \Rightarrow r = 1 \quad B = \{x \mid |x| < 1\}$

12) $z_1 z_3 = -3 - 11i, \; |z_3 - z_1| = 2\sqrt{2}, \; \frac{z_1}{z_2}\bar{z}_1 = 4 - 3i$

$z_3 = 5\, e^{-i(\arctan\frac{4}{3} + 2k\pi)}$

13 a) $z = -5 - 19i$ b) $z_0 = 1 + \sqrt{3}i, z_1 = iz_0 = -\sqrt{3} + i, z_2 = -1 - \sqrt{3}i, z_3 = \sqrt{3} - i$

c) $z = 1 - i, -1 - i, \pm\sqrt{2}i$ d) $z = \pm e^{-k\pi}, k \in \mathbb{Z}$

14) a) $A^T B = \begin{pmatrix} 7 & 4 & 6 \\ 25 & 3 & -1 \\ -15 & -1 & 2 \\ -20 & 0 & 5 \end{pmatrix}$ $\quad (B)A = \begin{pmatrix} 9 & 17 & -9 & -10 \\ 7 & 7 & -4 & -5 \\ 3 & 3 & -1 & 0 \end{pmatrix}$

b) $|B| = 0 \quad |C| = (1-t^2)(t+2) \quad , |AA^T| = 11$

c) $B\vec{x} = \vec{b}$: $\vec{x} = (t, 3-7t, -5+4t)$ bzw. keine Lösung

$A\vec{x} = \vec{b}$: $\vec{x} = (1, -4-t, -8-3t, t)$ bzw. $(2, 6-t, 11-3t, t)$

d) α) $t \in \mathbb{R} \setminus \{1, -1, -2\}$ $\quad \beta$) $t = 1; -2$ $\quad \gamma$) $t = -1$

15 a) $A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -8 & -5 \\ -1 & -4 & -3 \\ -1 & 2 & 1 \end{pmatrix}$

b) $|B| = t(t-2)(t+2)$ $\Rightarrow B\vec{x} = \vec{0}$ nichttrivial lösbar $\Leftrightarrow |B| = 0$

$\Leftrightarrow t = 0; \pm 2$

$t=0$ $\vec{x} = s(-1,1,1)$, $t=2$ $\vec{x} = s(-1,3,5)$, $t=-2$ $\vec{x} = s(-1,3,1)$

c) $\vec{x} = (4, 0, -1)$

16) Eigenwerte $|A - \lambda E| = \begin{vmatrix} 11-\lambda & 2 & -8 \\ 2 & 2-\lambda & 10 \\ -8 & 10 & 5-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 + 81\lambda - 1458 = 0$

$\Rightarrow \lambda_1 = 9, \lambda_2 = -9 \quad \lambda_3 = 18$

Eigenvektoren (Räume) $\quad (A - \lambda_i E)\vec{a} = \vec{0}$

$\lambda_1 = 9$: $\vec{a}_1 = t(2, 2, 1)$ $\quad \lambda_2 = -9$: $a_2 = (1, -2, 2)$

$\lambda_3 = 18$: $a_3 = t(-2, 1, 2)$