19) a) Niveaulinien $x^2-4y^2+1=c$ $\quad c\in\mathbb{R} \Rightarrow 4y^2 = x^2+1-c$

$\Rightarrow \quad y^2 = \frac{1}{4}(x^2+1-c) \quad y = \pm\frac{1}{2}\sqrt{x^2+1-c}$ mit $x^2 \geq c-1$

Für $c=1$ $\quad y = (\pm)\frac{1}{2}|x|$ d.h. $y = (\pm)x$

(Für $c \neq 1$ Hyperbeln).

b) Tangentialebene: $z - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$

$f_x = \frac{\partial}{\partial x} f = 2x \quad \frac{\partial}{\partial y} f = f_y = -8y \quad$ grad $f = (2x,-8y)$

$f(2,-1) = 9 \quad f_x(2,-1) = 4 \quad f_y(2,-1) = 8 \Rightarrow$ grad $f_{(2,-1)} = 4(1,2)$

Tangentialebene $z - 9 - 4(x-2) + 8(y+1) = 0$

Richtungsableitung: $\frac{\partial}{\partial \vec{a}} f|_{(2,-1)} = \text{grad} f(2,-1) \frac{\vec{a}}{\|a\|}$

$= \frac{4}{\sqrt{5}}(1,2)(2,-1) = 0$

20) Kritische Stellen: $\frac{\partial}{\partial x} xy e^{-x^2-y^2} = y(1-2x^2)e^{-x^2-y^2} = 0$

$\wedge \frac{\partial}{\partial y} xy e^{-x^2-y^2} = x(1-2y^2)e^{-x^2-y^2} = 0$

$\Rightarrow$ (I) $x(1-2y^2) = 0 \wedge$ (II) $y(1-2x^2) = 0$

Aus (I) folgt $x = 0 \vee (1-2y^2) = 0$

Aus $x = 0 \Rightarrow$ (II) $y = 0 \Rightarrow$ krit. Punkt $(0,0)$

Ist $x \neq 0 \Rightarrow$ (I) $1-2y^2 = 0$, also $y = \pm\frac{1}{\sqrt{2}}$, dann folgt aus

II $\quad 1-2x^2 = 0$, also $x = \pm\frac{1}{\sqrt{2}} \Rightarrow$ Kritische Punkte

$(\pm 1, \pm 1)\frac{1}{\sqrt{2}}$

Die Stelle $(0,0)$ ist ein Sattelpunkt, da $f(x,y) > 0$ für $x,y > 0$, $f(x,y) < 0$ für $x > 0, y < 0$, $f(0,0) = 0$.

Da $\lim_{x,y\to\infty} f(x,y) = 0$, besitzt $f$ sowohl ein absolutes Maximum, diese liegen in den kritischen Stelle.

wegen $f(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}) > 0$ und $f(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) < 0$

$\Rightarrow$ Max, bei $\pm(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, Min bei $\pm(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$.

Mit der Hesse-Matrix

$f_{xx} = 2xy(2x^2-3)e^{-x^2-y^2} \quad f_{yy} = 2xy(2y^2-3)e^{-x^2-y^2}$

$f_{xy} = f_{yx} = (1-2x^2)(1-2y^2)e^{-x^2-y^2}$

$\Rightarrow \Delta(x,y) = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix} = e^{-2x^2-2y^2}\left[4x^2y^2(2x^2-3)(2y^2-3) - (1-2x^2)^2(1-2y^2)^2\right]$

$\Delta(0,0) = -1 \Rightarrow (0,0)$ Sattelpunkt

$\Delta(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}) = 4e^{-1} > 0$, $f_{xx}(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = f_{xx}(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}) = -2e^{-1} < 0$

$\Rightarrow$ bei $\pm(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ Maximum $\quad f_{xx}(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}) = f_{xx}(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = 2e^{-1} > 0$

$\Rightarrow$ Minimum bei $\pm(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$ wobei

noch $f(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, $f(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = f(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$ gilt.